Anisotropic simplicial minisuperspace model in the presence of a scalar field

Cristóvão Correia da Silva [1] and *Ruth M. Williams* [2]

DAMTP, Silver Street, Cambridge,

CB3 9EW, England

ABSTRACT

We study an anisotropic simplicial minisuperspace model with a cosmological constant and a massive scalar field. After obtaining the classical solutions we then compute the semi-classical approximation of the no-boundary wave function of the Universe along a steepest descents contour passing through classical Lorentzian solutions. The oscillatory behaviour of the resulting wave funtion is consistent with the prediction of classical Lorentzian spacetime for the late Universe.

[1] e-mail address : clbc2@damtp.cam.ac.uk

[2] e-mail address : rmw7@damtp.cam.ac.uk

1 Introduction

The Euclidean path integral formulation of quantum gravity is arguably the most promising of all non-stringy approaches to quantum cosmology. The canonical quantisation approach translated by the Wheeler-DeWitt equation does not possess the flexibility of the path integral formulation as it cannot account for topology changing processes. The Euclidean path integral formulation with its sum over geometries naturally allows these processes to occur as it considers a sum over different topologies. As important as how to obtain the dynamics of the quantum state of the Universe is to use the correct prescription for the boundary condition needed to obtain the wave function of the Universe. The most natural boundary condition, the Hartle-Hawking no boundary proposal,[1], requires that the sum over geometries be restricted to those four-geometries which have no boundary except for the prescribed three-geometry where the arguments of the wave function are defined. This is specially well adapted to the path integral formalism as it can be easily implemented in a sum over topologies context. However, even this proposal is plagued by at least two main problems. Firstly, the Euclidean gravitational action is not bounded from below which leads to the divergence of the Euclidean path integral. Secondly, there is no clear prescription for the correct integration contour to use. In [2], Hartle proposes the use of the steepest descents contour in the space of complex metrics as the solution to both problems. Furthermore by choosing the steepest descents contour passing through the classical solutions of the theory, he made it very likely that the path integral be dominated by classical four-geometries, i.e., solutions of Einstein's equations and stationary points of the path integral, as desired for any wave function that is intended to represent our current Universe. Note that in this view, the fact that an integration solely over real-valued Euclidean geometries does not yield a convergent result for the path integral, is actually a good thing, for such a path integral would never predict the oscillatory behaviour in the late Universe that traditionally represents classical Lorentzian space-time.

As it is clearly impossible to calculate the full wave function integrated over all possible metric degrees of freedom, it is usual to explore approximated models in which the infinite degrees of freedom are reduced to only a few. The loss of generality induced by this reduction is not as bad it might seem at first, if we remember the relative symmetry of the observed Universe. These models are generically called minisuperspace models, for they "exist" in small, few-dimensional, subspaces contained in the infinite-dimensional superspace which is the space of all three-metrics. Particularly useful among minisuperspace models are the the ones based on Regge calculus. Such simplicial minisuperspace models were introduced by Hartle [3]. In such models one typically takes the simplicial complex which models the topology of interest to be fixed and the square edge length assignments play the role of the metric degrees of freedom. The summation over edge lengths models the continuum integration over the metric tensor. This approach has several advantages. First by treating the four-geometry directly it is more adequate to deal with the Hartle-Hawking proposal, [1], (with its four-dimensional nature), than the usual $3+1$ ADM decomposition of space-time, where a careful study of how the four-geometry closes off at the beginning of the universe is essential. Second, by discretizing space-time the classical equations become algebraic which makes it easier to find classical solutions which are essential to the semiclassical approximation. Third, it offers the possibility of systematic improvement.

In [2], Hartle studied the simplest simplicial minisuperspace model, the α_4 triangulation of the three-sphere. He assumed isotropy which leads to all boundary edge lengths being equal. Later in [4], Birmingham generalised his work to isotropic triangulations of Lens spaces $L(p,1)$. Another model was considered by Louko and Tuckey in [5] working in a minisuperspace anisotropic 2+1-dimensional cosmological model. In [6], Furihata considered a more general anisotropic triangulation of the three-sphere. Finally, in [7], we generalised the isotropic models to incorporate all simplicial four-geometries that are cones over closed connected simplicial three-manifolds, i.e., simplicial four-conifolds, and for the first time introduced a massive scalar field. In all these cases it was possible to find a steepest descent contour and to prove that the resulting wave function shows oscillatory behaviour for large spatial geometries. We now propose to study an extension of Furihata's model to incorporate a matter sector. We investigate the existence of a steepest descent contour and the properties of the resulting wave function.

2 General Regge Formalism

A convenient way of defining an $n-$simplex is to specify the coordinates of its $(n+1)$ vertices, $\sigma = [0,1,2,...,n]$. By specifying the squared values of the lengths of the edges $[i,j]$, s_{ij}, we fix the simplicial metric on the simplex.

$$g_{ij}(s_k) = \frac{s_{0i} + s_{0j} - s_{ij}}{2} \tag{2.1}$$

where $i, j = 1, 2, ..n$.

So if we triangulate a smooth manifold M endowed with a metric $g_{\mu\nu}$ by a homeomorphic simplicial manifold $\mathcal{M}$, the metric information is transferred to the simplicial metric of that simplicial complex

$$g_{\mu\nu}(x) \longrightarrow g_{ij}(\{s_k\}) = \frac{s_{0i} + s_{0j} - s_{ij}}{2} \tag{2.2}$$

In the continuum framework the sum over metrics is implemented through a functional integral over the metric components $\{g_{\mu\nu}(x)\}$. In the simplicial framework the metric degrees of freedom are the squared edge lengths, and so the functional integral is replaced by a simple multiple integral over the values of the edge lengths. But not all edge lengths have equal standing. Only the ones associated with the interior of the simplicial complex get to be integrated over. The boundary edge lengths remain after the sum over metrics and become the arguments of the wavefunction of the universe.

$$\int Dg_{\mu\nu}(x) \longrightarrow \int D\{s_i\} = \prod \int d\mu(s_i) \tag{2.3}$$

In the simplicial framework the fact that the geometry of the complexes is completely fixed by the specification of the squared values of all edge lengths, means that all geometrical quantities, such as volumes and curvatures, can be expressed completely in terms of those edge lengths. Consequently the Regge action (the simplicial analogue of the Einstein action for GR) associated with a complex of known topology can be expressed exclusively in terms of those edge lengths.

The Euclideanized Einstein action for a smooth $4-$manifold M with boundary ∂M, and endowed with a $4-$metric, $g_{\mu\nu}$, and a scalar field Φ with mass m, is

$$\begin{aligned} I[M, g_{\mu\nu}, \phi] &= -\int_M d^4x\sqrt{g}\frac{(R-2\Lambda)}{16\pi G} - \int_{\partial M} d^3x\sqrt{h}\frac{K}{8\pi G} + \\ &+ \frac{1}{2}\int_M d^4x\sqrt{g}(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) \end{aligned}$$

where K is the extrinsic curvature.

Its simplicial analogue will be the Regge action for a combinatorial $4-$manifold, $\mathcal{M}$, with squared edge lengths $\{s_k\}$, and with a scalar field taking values $\{\phi_v\}$ for each vertex v of $\mathcal{M}$, [8]:

$$
\begin{aligned}
I[\mathcal{M},\{s_k\},\{\phi_v\}] &= \frac{-2}{16\pi G}\sum_{\sigma_2^i} V_2(\sigma_2^i)\theta(\sigma_2^i) + \frac{2\Lambda}{16\pi G}\sum_{\sigma_4} V_4(\sigma_4) \\
&- \frac{2}{16\pi G}\sum_{\sigma_2^b} V_2(\sigma_2^b)\psi(\sigma_2^b) + \frac{1}{2}\sum_{\sigma_1=[ij]} \tilde{V}_4(\sigma_1)\frac{(\phi_i-\phi_j)^2}{s_{ij}} \\
&+ \frac{1}{2}\sum_j \tilde{V}_4(j) m^2\phi_j^2
\end{aligned}
$$

where:

- σ_k denotes a $k-$simplex belonging to the set Σ_k of all $k-$simplices in $\mathcal{M}$.
- $\theta(\sigma_2^i)$, is the deficit angle associated with the interior $2-$simplex $\sigma_2^i = [ijk]$

$$\theta(\sigma_2^i) = 2\pi - \sum_{\sigma_4 \in St(\sigma_2^i)} \theta_d(\sigma_2^i, \sigma_4) \tag{2.4}$$

  and $\theta_d(\sigma_2^i,\sigma_4)$ is the dihedral angle between the $3-$simplices $\sigma_3 = [ijkl]$ and $\sigma_3^{'} = [ijkm]$, of $\sigma_4 = [ijklm]$ that intersect at σ_2^i. Its full expression is given by [4].

- $\psi(\sigma_2^b)$ is the deficit angle associated with the boundary $2-$simplex σ_2^b:

$$\psi(\sigma_2^b) = \pi - \sum_{\sigma_4 \in St(\sigma_2^b)} \theta_d(\sigma_2^b, \sigma_4) \tag{2.5}$$

- $V_k(\sigma_k)$ for $k = 2,3,4$ is the $k-$volume associated with the $k-$simplex, σ_k, and once again their explicit expressions in terms of the squared edge lengths are given by [4].
- $\tilde{V}_4(\sigma_1)$ is the $4-$volume in the simplicial complex $\mathcal{M}$, associated with the edge σ_1, i.e., the volume of the space occupied by all points of $\mathcal{M}$ that are closer to σ_1 than to any other edge of $\mathcal{M}$. The same holds for $\tilde{V}_4(j)$ where j represents all vertices of $\mathcal{M}$.

It is easy to see that both $\tilde{V}_4(\sigma_1)$ and $\tilde{V}_4(j)$, can be expressed exclusively in terms of the edge lengths $\{s_k\}$. All these expressions remain valid if we consider smooth conifolds and their combinatorial counterparts.

If we are working in a simplicial minisuperspace model based on a simplicial four-manifold $\mathcal{M}_4$, with boundary edges s_b, interior edges s_i, and in the presence of a scalar field taking values, ϕ_k, at the vertices $k \in \mathcal{M}_4$, then the wave function of the Universe is of the type:

$$\Psi[\partial\mathcal{M}, \{s_b\}, \{\phi_b\}] = \int D\{s_i\} D\{\phi_i\} e^{-I[\mathcal{M}^4, \{s_i\}, \{s_b\}, \{\phi_i\}, \{\phi_b\}]} \tag{2.6}$$

where

- $\{s_i\}$ are the squared lengths of the interior edges
- $\{s_b\}$ are the squared lengths of the boundary edges
- $\{\phi_i\}$ are the values of the field at the interior vertices
- $\{\phi_b\}$ are the values of the field at the boundary vertices

Although the functional integral over metrics has been written explicitly in terms of the edge lengths, this expression is still heuristic because we still need to specify the measure, and the integration contour to be used.

3 The Model

As in [6] our simplicial complex can be seen as a cone, $\mathcal{M}^4 = 0 * \mathcal{M}^3$, over an anisotropic triangulation of the three-sphere, $\mathcal{M}^3 = \alpha_4(a, b)$. See figure 1. There are 10 boundary edges, five with squared length, a, and the other five with squared length b. There are also five interior edges connecting the five boundary vertices $1, 2, 3, 4, 5$ to the interior vertex 0, and they all have squared length s_i. Note that although the model is anisotropic because there are two kinds of boundary edges, it is still homogeneous because all boundary vertices are connected to the same number and kind of edges. Furthermore, we consider the scalar field to take the same value, ϕ_b at all boundary vertices, and ϕ_i at the interior vertex 0.

For simplicity we will use new variables,

$$\xi = \frac{s_i}{a} \tag{3.1}$$

$$S = \frac{H^2 a}{l^2} \tag{3.2}$$

$$\beta = \frac{b}{a} \tag{3.3}$$

where $H^2 = l^2\Lambda/3$, and $l^2 = 16\pi G$ is the Planck length. We shall work in units where $c = \hbar = 1$.

The Euclidean action I, is then a function of $\xi, S, \beta, \phi_b, \phi_i$, and the minisuperspace wave function is

$$\Psi[S,\beta,\phi_b] = \int_C D\xi D\phi_i e^{-I[\xi,S,\beta,\phi_i,\phi_b]} \tag{3.4}$$

To calculate the explicit expression of the action we must compute the volumes and deficit angles associated with the various $p-simplices$ in $\mathcal{M}_4$.

There is only one kind of $4-simplex$, eg., [01234], and its volume is

$$V_4[01234] = \frac{1}{4!}\left(\frac{a}{2}\right)^2 \sqrt{-\beta^2+3\beta-1}\sqrt{4\xi(\beta+1)-1-\beta-\beta^2} \tag{3.5}$$

There is also only one kind of boundary tetrahedron, eg., [1234], and its volume is

$$V_3^i[1234] = \sqrt{\frac{2}{(3!)^2}\left(\frac{a}{2}\right)^3 (\beta+1)(-\beta^2+3\beta-1)} \tag{3.6}$$

The existence of two different boundary edges means that there are two kinds of interior tetrahedra

$$V_{3A}^b[0123] = \left(\frac{1}{3!}\right)\left(\frac{a}{2}\right)^{3/2} \sqrt{2\beta\{(4-\beta)\xi-1\}} \tag{3.7}$$

$$V_{3B}^b[0124] = \left(\frac{1}{3!}\right)\left(\frac{a}{2}\right)^{3/2} \sqrt{2\{(4\beta-1)\xi-\beta^2\}} \tag{3.8}$$

There are also two kinds of interior triangles, whose areas are

$$V_{2A}^i[012] = \frac{a}{2}\sqrt{\xi-1/4} \tag{3.9}$$

$$V_{2B}^i[013] = \frac{a\sqrt{\beta}}{2}\left(\xi-\frac{\beta}{4}\right)^{1/2} \tag{3.10}$$

Finally there are also two kinds of boundary triangles

$$V_{2A}^b[124] = \frac{a}{2}\sqrt{\beta-1/4} \tag{3.11}$$

$$V_{2B}^b[123] = \frac{a\sqrt{\beta}}{2}\left(1-\frac{\beta}{4}\right)^{1/2} \tag{3.12}$$

The $4-volumes$ associated with each boundary vertex b, and with the interior vertex 0 are

$$\tilde{V}_4[b] = \frac{4}{5} V_4[01234], \qquad \tilde{V}_4[0] = V_4[01234] \tag{3.13}$$

The $4-volume$ associated with each interior edge $[0,b]$ is

$$\tilde{V}_4[0,b] = \frac{2}{5} V_4[01234] \tag{3.14}$$

The deficit angles associated with each kind of triangle are

$$\theta[012] = 2\pi - \arccos h_1 - 2\arccos h_2 \tag{3.15}$$

$$\theta[013] = 2\pi - \arccos h_3 - 2\arccos h_4 \tag{3.16}$$

$$\psi[123] = \pi - 2\arccos h_5 \tag{3.17}$$

$$\psi[124] = \pi - 2\arccos h_6 \tag{3.18}$$

where the dihedral angles are

$$h_1 = \frac{2(\beta^2 - 2\beta + 2)\xi - \beta^2 + \beta - 1}{2\beta(4-\beta)(\xi - \xi_{3A})} \tag{3.19}$$

$$h_2 = \frac{2(3\beta - 2)\xi - \beta^2 - \beta + 1}{2\sqrt{\beta(4\beta-1)(4-\beta)}\sqrt{(\xi-\xi_{3A})(\xi-\xi_{3B})}} \tag{3.20}$$

$$h_3 = \frac{2(2\beta^2 - 2\beta + 1)\xi - \beta(\beta^2 - \beta + 1)}{2(4\beta - 1)(\xi - \xi_{3B})} \tag{3.21}$$

$$h_4 = \sqrt{\frac{\beta}{(4-\beta)(4\beta-1)}} \frac{2(3-2\beta)\xi + \beta^2 - \beta - 1}{2\sqrt{(\xi-\xi_{3A})(\xi-\xi_{3B})}} \tag{3.22}$$

$$h_5 = \frac{1}{2}\sqrt{\frac{-\beta^2 + 3\beta - 1}{(4\beta-1)(\beta+1)}} \frac{1}{\sqrt{\xi - \xi_{3B}}} \tag{3.23}$$

$$h_6 = \frac{1}{2}\sqrt{\frac{\beta(-\beta^2 + 3\beta - 1)}{(4-\beta)(\beta+1)}} \frac{1}{\sqrt{\xi - \xi_{3A}}} \tag{3.24}$$

where

$$\xi_{3A} = \frac{1}{4-\beta} \qquad \xi_{3B} = \frac{\beta^2}{4\beta - 1} \tag{3.25}$$

are the values of ξ for which the interior tetrahedra become degenerate.

The Regge minisuperspace action then becomes

$$
\begin{aligned}
I[\xi, S, \beta, \phi_i, \phi_b] \;\; = \;\; & -\left(\frac{S}{H^2}\right)\left\{\frac{5}{2}\sqrt{4\beta-\beta^2}(\pi - 2\arccos h_6) + \frac{5}{2}\sqrt{4\beta-1}(\pi - 2\arccos h_5)\right. \\
+ \;\; & 5\sqrt{\xi - 1/4}(2\pi - \arccos h_1 - 2\arccos h_2) \\
+ \;\; & 5\sqrt{\beta}\sqrt{\xi - \frac{\beta}{4}}(2\pi - \arccos h_3 - 2\arccos h_4) \\
- \;\; & \left.\frac{1}{48}\sqrt{\beta+1}\sqrt{-\beta^2+3\beta-1}\frac{\sqrt{\xi-\xi_4}}{\xi}(\phi_i-\phi_b)^2 l^2\right\} \\
+ \;\; & \left(\frac{S}{H}\right)^2\left(\frac{5}{8}\right)\sqrt{\beta+1}\sqrt{-\beta^2+3\beta-1}\sqrt{\xi-\xi_4}\left[1+\frac{1}{60}\left(\frac{ml}{H}\right)^2(4\phi_b^2 l^2 + \phi_i^2 l^2)\right]
\end{aligned}
\tag{3.26}
$$

where

$$
\xi_4 = \frac{\beta^2+\beta+1}{4\beta+4} \tag{3.27}
$$

is the the value of ξ for which the $4-$*simplices* become degenerate.

Since the wave function is to be obtained as an integral over ξ and ϕ_i, it is essential we study, term by term, the analytic and asymptotic properties of the action as a function of these variables. It is easy to see that the action is an analytic function for all values of ϕ_i. However, I as a function of ξ has several square root branch points, and logarithmic branching points and infinities.

Note that a term $\arccos u(z)$ has branch points at $u(z) = +1, -1$, and at $u(z) = \infty$. The associated branch cuts are usually taken to be $(-\infty, -1] \cup [1, +\infty)$. Since

$$
\arccos u(z) = -i \log\left(u(z) + \sqrt{u(z)^2 - 1}\right)
$$

then we see that there are logarithmic singularities when $u(z) = \infty$. The table below shows the logarithmic branching points and infinities associated with the dihedral angles. Furthermore, associated with the square root terms we have branch points at $\xi = \frac{1}{4}$, $\xi = \frac{\beta}{4}$, and $\xi = \xi_4$.

Dihedral angles	-1	$+1$	∞
$h_1(\xi)$	ξ_4	$1/4$	ξ_{3A}
$h_2(\xi)$	$1/4, \xi_4$	$1/4, \xi_4$	ξ_{3A}, ξ_{3B}
$h_3(\xi)$	ξ_4	$\beta/4$	ξ_{3B}
$h_4(\xi)$	$\beta/4, \xi_4$	$\beta/4, \xi_4$	ξ_{3A}, ξ_{3B}
$h_5(\xi)$	ξ_4	ξ_4	ξ_{3B}
$h_6(\xi)$	ξ_4	ξ_4	ξ_{3A}

Implementing the necessary branch cuts it is easy to conclude that the Riemann surface of the action is composed of an infinite number of sheets, each with a branch cut $(-\infty, \xi_4]$. However, as in [7], only three sheets will be relevant for the computation of the path integral. We define the first sheet $\mathbf{C}_1$ of $I[\xi]$ as the sheet where the terms in $\arccos(z)$ assume their principal values. So the action in the first sheet will be formally equal to the original expression (3.26). Note that with the first sheet defined in this way, for real $\xi > \xi_4$ the volumes and deficit angles are all real, leading to a real Euclidean action for $\xi \in [\xi_4, +\infty)$ on the first sheet. On the other hand, when ξ is real and less than $min(1/4, \beta/4)$ in the first sheet , the volumes become pure imaginary and the Euclidean action becomes pure imaginary. For all other points of this first sheet the action is fully complex.

Note that since we are only interested in geometries in which the boundary three-metric is positive definite, we must require that the volume of the boundary three-simplices be positive, which is equivalent to requiring $\frac{3-\sqrt{5}}{2} < \beta < \frac{3+\sqrt{5}}{2}$. Furthermore, since the simplicial metric in each four-simplex is real if and only if ξ is real, then the simplicial geometries built out of these $4-$simplices will be real only when ξ is real. Finally, computing the eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ of the simplicial metric, g_{ij}, we obtain

$$\lambda_1 = \frac{1}{4}\left[8\xi - (\beta+1) - \sqrt{[8\xi - 2(\beta+1)]^2 + (\beta-1)^2}\right] \tag{3.28}$$

$$\lambda_2 = \frac{1}{4}\left[8\xi - (\beta+1) + \sqrt{[8\xi - 2(\beta+1)]^2 + (\beta-1)^2}\right] \tag{3.29}$$

$$\lambda_3 = \frac{1}{4}[1 + \beta + \sqrt{5}(\beta - 1)] \tag{3.30}$$

$$\lambda_4 = \frac{1}{4}[1 + \beta - \sqrt{5}(\beta - 1)] \tag{3.31}$$

It is easy to see that since $\frac{3-\sqrt{5}}{2} < \beta < \frac{3+\sqrt{5}}{2}$, then $\lambda_2, \lambda_3, \lambda_4$, are all positive and so the signature is Euclidean $(++++)$, when $\lambda_1 > 0$, i.e., when $\xi > \xi_4$ and Lorentzian $(-+++)$, when $\lambda_1 < 0$, i.e., when $\xi < \xi_4$. So we see that for real $\xi > \xi_4$ we have real Euclidean signature geometries, with real Euclidean action, and for real $\xi < min(\frac{1}{4}, \frac{\beta}{4})$, we have real Lorentzian signature geometries with pure imaginary Euclidean action.

When the action is continued in ξ once around all finite branch points $\xi = 1/4, \beta/4, \xi_{3A}, \xi_{3B}$, and ξ_4, we reach what shall be called the second sheet . It is easy to conclude that the action in this second sheet is just the negative of the action in the first sheet.

$$I^I[\xi, \beta, S, \phi_i, \phi_b] = -I^{II}[\xi, \beta, S, \phi_i, \phi_b] \tag{3.32}$$

Once in the second sheet, if we encircle the branch points in such a way that we cross the branch cut, $(-\infty, \xi_4]$, between $1/4$ and $\beta/4$, we arrive at what we shall call the third sheet. By doing this the terms $\arccos h_1$, and $\arccos h_2$, change signs, but $\sqrt{\xi - 1/4}$ does not and so the action in this third sheet is different from in the first sheet

$$\begin{aligned} I_{III}[\xi, S, \beta, \phi_i, \phi_b] \; = - \; & \left(\frac{S}{H^2}\right)\Big\{\frac{5}{2}\sqrt{4\beta - \beta^2}(\pi - 2\arccos h_6) + \frac{5}{2}\sqrt{4\beta - 1}(\pi - 2\arccos h_5) \\ - \; & 5\sqrt{\xi - 1/4}(2\pi + \arccos h_1 + 2\arccos h_2) \\ + \; & 5\sqrt{\beta}\sqrt{\xi - \frac{\beta}{4}}(2\pi - \arccos h_3 - 2\arccos h_4) \\ - \; & \frac{1}{48}\sqrt{\beta + 1}\sqrt{-\beta^2 + 3\beta - 1}\frac{\sqrt{\xi - \xi_4}}{\xi}(\phi_i - \phi_b)^2 l^2\Big\} \\ + \; & \left(\frac{S}{H}\right)^2\left(\frac{5}{8}\right)\sqrt{\beta + 1}\sqrt{-\beta^2 + 3\beta - 1}\sqrt{\xi - \xi_4}\Big[1 + \frac{1}{60}\left(\frac{ml}{H}\right)^2(4\phi_b^2 l^2 + \phi_i^2 l^2)\Big] \end{aligned} \tag{3.33}$$

3.1 Asymptotic Behaviour of the Action

In any discussion of the convergence of an integral over an infinite contour the asymptotic behaviour of the integrand, when $|\xi| \to \infty$, is essential.

In the first sheet when $\xi \to \infty$ the action behaves like

$$\begin{aligned} I^I[\xi, \beta, S, \phi_i, \phi_b] \; \sim \; & \frac{5}{8}\sqrt{(\beta + 1)(-\beta^2 + 3\beta - 1)} \\ \times \; & \Big[1 + \frac{1}{60}\frac{m^2 l^2}{H^2}(4\phi_b^2 l^2 + \phi_i^2 l^2)\Big]\frac{S}{H^2}(S - S^I_{crit})\sqrt{\xi} \end{aligned} \tag{3.34}$$

where

$$
\begin{aligned}
S^I_{crit} &= \frac{8}{\sqrt{(\beta+1)(-\beta^2+3\beta-1)}}\Big[(2\pi - \arccos h_1^\infty - 2\arccos h_2^\infty) \\
&+ \sqrt{\beta}(2\pi - \arccos h_3^\infty - 2\arccos h_4^\infty)\Big]\left[1+\frac{1}{60}\frac{m^2l^2}{H^2}(4\phi_b^2l^2+\phi_i^2l^2)\right]^{-1}
\end{aligned}
\tag{3.35}
$$

and

$$
h_1^\infty = \frac{\beta^2-2\beta+2}{\beta(4-\beta)} \tag{3.36}
$$

$$
h_2^\infty = \frac{3\beta-2}{\sqrt{\beta(4\beta-1)(4-\beta)}} \tag{3.37}
$$

$$
h_3^\infty = \frac{2\beta^2-2\beta+1}{4\beta-1} \tag{3.38}
$$

$$
h_4^\infty = \sqrt{\frac{\beta}{(4-\beta)(4\beta-1)}}(3-2\beta) \tag{3.39}
$$

For the second sheet the asymptotic behaviour of the action is just the negative of that in the first sheet. For the third sheet we have

$$
\begin{aligned}
I^{III}[\xi,\beta,S,\phi_i,\phi_b] &\sim \frac{5}{8}\sqrt{(\beta+1)(-\beta^2+3\beta-1)} \\
&\times \left[1+\frac{1}{60}\frac{m^2l^2}{H^2}(4\phi_b^2l^2+\phi_i^2l^2)\right]\frac{S}{H^2}(S+S^{III}_{crit})\sqrt{\xi}
\end{aligned}
\tag{3.40}
$$

where

$$
\begin{aligned}
S^{III}_{crit} &= \frac{8}{\sqrt{(\beta+1)(-\beta^2+3\beta-1)}}\Big[(2\pi + \arccos h_1^\infty + 2\arccos h_2^\infty) \\
&- \sqrt{\beta}(2\pi - \arccos h_3^\infty - 2\arccos h_4^\infty)\Big]\left[1+\frac{1}{60}\frac{m^2l^2}{H^2}(4\phi_b^2l^2+\phi_i^2l^2)\right]^{-1}
\end{aligned}
\tag{3.41}
$$

4 Classical Solutions

The classical simplicial geometries are the extrema of the Regge action we obtained above. In our minisuperspace model there are two degrees of freedom ξ, ϕ_i. So the Regge equations of motion will be:

$$\frac{\partial I}{\partial \xi} = 0 \tag{4.42}$$

and

$$\frac{\partial I}{\partial \phi_i} = 0 \tag{4.43}$$

They are to be solved for the values of ξ, ϕ_i, subject to the fixed boundary data S, β, ϕ_b. The classical solutions will thus be of the form $\overline{\xi}(S, \beta, \phi_b)$, and $\overline{\phi}_i(S, \beta, \phi_b)$. The solution $\overline{\xi}(S, \beta, \phi_b)$ completely determines the simplicial geometry.

Note that we shall be working on the first sheet. Of course, since on the second sheet the action is just the negative of this, the equations of motion are the same. And obviously every classical solution $\overline{\xi}_I(S, \beta, \phi_b)$ located on the first sheet will have a counterpart $\overline{\xi}_{II}$ of the same numerical value, but located on the second sheet, and so with an action of opposite sign, $I[\overline{\xi}_I(S, \beta, \phi_b)] = -I[\overline{\xi}_{II}(S, \beta, \phi_b)]$. So the classical solutions occur in pairs.

Inserting the second equation, (4.43),

$$\phi_i = \frac{\phi_b}{1 + \frac{1}{2}\frac{m^2 l^2}{H^2}\xi S} \tag{4.44}$$

in the first equation (4.42), we obtain a very long cubic equation on S for each value of ξ, given fixed β and ϕ_b. This equation can then be solved numerically for ξ, and by inverting the resulting solutions we obtain three branches of solutions $\xi = \xi_{cl}(S, \beta, \phi_b)$. For obvious physical reasons we shall accept only solutions with real positive S. In figure 2 we show one such solution for a fixed value of β and ϕ_b.

The general form of this solution is similar to the ones associated with other values of β and ϕ_b. We see that in general as $\xi \to \xi_0 = min(1/4, \beta/4)$, the value of S diverges to $+\infty$. It is this branch that will represent the classical solutions for the late Universe. The fact that these solutions are Lorentzian means that semiclassically the wave function of the Universe should be dominated by the contribution coming from classical Lorentzian spacetimes like our own, as desired.

As in previous cases the domain of solutions is divided by the line $S = S^I_{crit}(cl)$, where

$$
\begin{aligned}
S^I_{crit}(cl) &= S^I_{crit}(\phi_i = \phi_i^{cl}) = \frac{8}{\sqrt{(\beta+1)(-\beta^2+3\beta-1)}}\Big[(2\pi - \arccos h_1^\infty \\
&- 2\arccos h_2^\infty) + \sqrt{\beta}(2\pi - \arccos h_3^\infty - 2\arccos h_4^\infty)\Big]\left(1 + \frac{2}{15}K\phi_b^2\right)^{-1}
\end{aligned}
$$

where $K = \frac{m^2 l^2}{2H^2}$.

We see that for $0 < S < S^I_{crit}(cl)$, we will have:

- Two pairs of real Lorentzian signature solutions $\overline{\xi}_I^{L1}(S,\beta,\phi_b) = \overline{\xi}_{II}^{L1}(S,\beta,\phi_b) \in (-\infty, \xi_0]$, and $\overline{\xi}_I^{L2}(S,\beta,\phi_b) = \overline{\xi}_{II}^{L2}(S,\beta,\phi_b) \in (-\infty, \xi_0]$ with pure imaginary Euclidean actions.
- One pair of real Euclidean signature solutions $\overline{\xi}_I^{E}(S,\beta,\phi_b) = \overline{\xi}_{II}^{E}(S,\beta,\phi_b) \in [\xi_4, +\infty)$, with real Euclidean action.

For $S > S^I_{crit}(cl)$ we have:

- Only one pair of real solutions $\overline{\xi}_I(S,\beta,\phi_b) = \overline{\xi}_{II}(S,\beta.\phi_b) \in (-\infty, \xi_0]$ that correspond to Lorentzian signature simplicial metrics, and whose Euclidean actions, though symmetric, are both pure imaginary.

$$
I[\overline{\xi}_I(S,\beta,\phi_b)] = -I[\overline{\xi}_{II}(S,\beta,\phi_b)] = i\tilde{I}[\overline{\xi}_I(S,\beta\phi_b)]
$$

If we increase the value of m or ϕ_b, the value of S_{crit} decreases to zero and eventually the branch associated with the Euclidean regime vanishes. Furthermore we can see that $S^I_{crit}(cl)$ as a function of the anisotropy β, becomes infinite at the points of maximum anisotropy. See figure 3.

5 Steepest Descent Contour

After studying the analytical and asymptotic properties of the action we can now focus on the Euclidean path integral that yields the wave function of the Universe.

$$
\Psi[S,\beta,\phi_b] = \int_C D\xi D\phi_i e^{-I[\xi,S,\beta,\phi_i,\phi_b]} \tag{5.45}
$$

In our simplified models the result obtained from a contour C is not very sensitive to the choice of measure if we stick to the usual measures, i.e., polynomials of the squared edge lengths. So we shall take

$$D\xi D\phi_i = \frac{ds_i}{2\pi i l^2} d\phi_i = \frac{S}{2\pi i H^2} d\xi d\phi_i \tag{5.46}$$

As we have mentioned above there is as yet no universally accepted prescription for the integration contour to use in quantum cosmology. Following Hartle [2], we shall accept that the main criteria any contour should satisfy are that it should lead to a convergent path integral and to a wave function predicting classical Lorentzian spacetime in the late Universe. The steepest descents contour over complex metrics seems to be the leading candidate. In the simplicial framework, complex metrics arise from complex-valued squared edge lengths, (2.1). The boundary squared edge lengths, S, and β have to be real and positive for obvious physical reasons. But the interior squared edge length, ξ, can be allowed to take complex values.

In general, a SD contour associated with an extremum ends up either at ∞, at a singular point of the integrand, or at another extremum with the same value of $Im(I)$. We have seen that when S is big enough the only classical solutions are a pair of real Lorentzian solutions $(\overline{\xi}_I(S, \beta, \phi_b), \overline{\phi}_i(S, \beta, \phi_b))$ and $(\overline{\xi}_{II}(S, \beta, \phi_b), \overline{\phi}_i(S, \beta, \phi_b))$, where $\overline{\xi}_I = \overline{\xi}_{II} < min(1/4, \beta/4)$. They are located on the first and second sheets respectively, and so have pure imaginary actions of opposite sign. Given that their actions are different valued no single SD path can go directly from one to the other extremum. On the other hand given that

$$I[\overline{\xi}] = [I[\overline{\xi}^*]]^*$$

and

$$I[\overline{\xi}_I] = -I[\overline{\xi}_{II}]$$

where $*$ denotes complex conjugation, we see that the SD path that passes through $\overline{\xi}_{II}$ will be the complex conjugate of the SD path that passes through $\overline{\xi}_I$. So the total SD contour will always be composed of two complex conjugate sections, each passing through one extremum, and this together with the real analyticity of the action guarantees that the resulting wavefunction is real.

The SD contour passing through the classical solution $\{\xi_{cl}(S, \beta, \phi_b), \phi_i^{cl}(S, \beta, \phi_b)\}$ is

$$C_{SD}(S,\beta,\phi_b) = \Big\{(\xi \in R, \phi_i) : Im[I(S,\xi,\beta,\phi_i,\phi_b)] = \tilde{I}[\xi_{cl}(S,\beta,\phi_b), \phi_i^{cl}(S,\beta,\phi_b)]\Big\} \quad (5.47)$$

where R is the Riemann sheet of the action, and $\tilde{I}(\xi) = iI(\xi)$.

In figure 4 we show the result of a numerical computation of this contour for $m = 1$, $\phi_b = 1$, $\beta = 1.5$ and $S = 50$.

The behaviour is similar to that for other values of the above variables. Going upward from the extremum, the SD contour proceeds to infinity in the first quadrant along the curve

$$\begin{aligned} & \frac{5}{8}\sqrt{(\beta+1)(-\beta^2+3\beta-1)}\Big[1+\frac{1}{60}\frac{m^2l^2}{H^2}(4\phi_b^2l^2+\phi_i^2l^2)\Big] \\ \times \quad & \frac{S}{H^2}(S-S^I_{crit})Im(\sqrt{\xi}) = \tilde{I}[\xi_{cl},\phi_{cl}] \end{aligned}$$

The convergence of the integral along this part of the contour for any polynomial measure is guaranteed by the asymptotic behaviour of the action on the first sheet

$$\begin{aligned} Re[I^I(\xi,\beta,S,\phi_i,\phi_b)] \quad \sim \quad & \frac{5}{8}\sqrt{(\beta+1)(-\beta^2+3\beta-1)}\Big[1+\frac{1}{60}\frac{m^2l^2}{H^2}(4\phi_b^2l^2+\phi_i^2l^2)\Big] \\ \times \quad & \frac{S}{H^2}(S-S^I_{crit})\sqrt{|\ \xi\ |} \end{aligned}$$

As we move downward from the classical solution we immediately cross the branch cut and the contour enters the second sheet. The contour then proceeds to cross the branch cut once more, this time between $1/4$ and $\beta/4$, emerging onto the third sheet where it finally proceeds to infinity inside the first quadrant along the curve

$$\begin{aligned} & \frac{5}{8}\sqrt{(\beta+1)(-\beta^2+3\beta-1)}\Big[1+\frac{1}{60}\frac{m^2l^2}{H^2}(4\phi_b^2l^2+\phi_i^2l^2)\Big] \\ \times \quad & \frac{S}{H^2}(S+S^{III}_{crit})Im(\sqrt{\xi}) = \tilde{I}[\xi_{cl},\phi_{cl}] \end{aligned}$$

As on the first sheet the convergence of the path integral along this section of the contour is guaranteed by the asymptotic behaviour of the real part of the Euclidean action along it

$$\begin{aligned} Re[I^{III}(\xi,\beta,S,\phi_i,\phi_b)] \quad \sim \quad & \frac{5}{8}\sqrt{(\beta+1)(-\beta^2+3\beta-1)}\Big[1+\frac{1}{60}\frac{m^2l^2}{H^2}(4\phi_b^2l^2+\phi_i^2l^2)\Big] \\ \times \quad & \frac{S}{H^2}(S+S^{III}_{crit})\sqrt{|\ \xi\ |} \end{aligned}$$

5.1 Semiclassical Approximation

From our study of the SD contour for $S > S_{crit}$, we can conclude that the range of integration in the neighbourhoods of the classical Lorentzian solutions give the dominant contribution to the integral, since the contribution of the other critical points, i.e., the infinities, is negligible given the asymptotic behaviour of the action. So in order to obtain the relevant information about the wave function of the Universe it is not really necessary to do the full computation of the integral along the SD contour. A semiclassical approximation based on the classical solutions found above will suffice.

Since for $S > S_{crit}$ these are real Lorentzian solutions with purely imaginary actions $I_k = i\tilde{I}[\xi_k^{cl}(S,\beta,\phi_b), \phi_i^{cl}(S,\beta,\phi_b)]$, then for $S > S_{crit}$, the semiclassical approximation is

$$\begin{aligned}\Psi_{SC}(S,\phi_b) &\sim \sum_{k=I,II}\sqrt{\frac{S^2}{2\pi H^4 det\{\frac{\partial^2\tilde{I}}{\partial x_i \partial x_j}\}}}e^{-i[\tilde{I}_k(S,\beta,\phi_b)-\frac{\pi}{4}]} \\ &\sim \sqrt{\frac{S^2}{2\pi H^4 det\{\frac{\partial^2\tilde{I}}{\partial x_i \partial x_j}\}}}2\cos\left[\tilde{I}_{cl}(S,\beta,\phi_b)-\frac{\pi}{4}\right]\end{aligned}$$

This semiclassical approximation is specially good when the integrand is sharply peaked about the classical solutions (extrema), which is particularly true when the argument of the exponential is large. This will be the case for the large S of the late Universe and for the whole range of S when $H^2 = \Lambda l^2/3$ is sufficiently small as it is the case of our late Universe.

In figure 5, we show the result of the numerical computation of one such semiclassical approximation. It is clear that the wave function exhibits the oscillatory behaviour that characterises the prediction of classical Lorentzian spacetime in the late Universe as desired.

For $S < S_{crit}$ the situation is not so simple as there are two pairs of classical Lorentzian solutions as well as the usual pair of Euclidean solutions. A semiclassical approximation could conceivably be based on any of these pairs. Furthermore, as we increase the values of ϕ_b and m, the value of S_{crit} sharply decreases and so we can envisage a situation where the range of existence of these solutions practically vanishes.

Computing the wave funtion for $S = S^I_{crit}$, (figure 6), we see that as in the pure gravity case , [6], the wave function peaks for universes with large anisotropy, and small values of the scalar field. If as in the continuum case we consider a scenario of quantum nucleation of the Lorentzian universe at $S = S^I_{crit}$, this result seems to favour the universes with larger anisotropy.

6 Conclusions

We have found that the results obtained by us in [7] can be extended to anisotropic models. In particular, following [6], we have considered the simplicial minisuperspace based on the cone over the simplest anisotropic triangulation of the three-sphere, coupled to a massive scalar field, ϕ_k. The anisotropy is reflected in the existence of two different kinds of boundary edge lengths. However, since we admit only one kind of internal edge length, the minisuperspace is still two-dimensional as in the isotropic models considered in [7]. This means that the path integral will not involve any new variables. We have found that for the late Universe the only classical solutions are pairs of real Lorentzian spacetimes like our own. We showed not only that there is a steepest descents contour going through these classical solutions but also that it yields a convergent path integral. These SD contours are very similar to those obtained in [7], but their behaviour is slightly more complex due to the existence of a larger number of branch points in the action function. As in the isotropic models, the semiclassical approximation is quite good, specially when S is large or when H is sufficiently small, as it is in our late Universe. The computation of the semiclassical wavefunction shows its oscillatory behaviour, characteristic of the prediction of classical Lorentzian spacetimes, as desired.

Combined with the results in [7] and [6] , the results of this paper show not only the versatility of the simplicial approach to minisuperspace models, but also that their predictions are generically in agreement with the results from similar continuum models. In all of them we find that by making the natural choice of the integration contour as the SD contour, the wave functions all predict classical Lorentzian spacetimes for the late Universe.

Acknowledgements

The work of C. Correia da Silva was supported by the Portuguese Ministry of Science and Technology under grant PRAXIS XXI BD-5905/95. Other support has come from the UK Particle Physics and Astronomy Research Council.

References

[1] J.B.Hartle and S.W.Hawking, Phys.Rev.D**28** (1983) 2960.

[2] J.B.Hartle, J.Math.Phys **30** (1989) 452.

[3] J.B.Hartle, J.Math.Phys **26** (1985) 804.

[4] D.Birmingham, Phys.Rev.D**52** (1995) 5760.

[5] J.Louko and P.A.Tuckey, Class.Quantum Grav.**9** (1992) 41.

[6] Y.Furihata, Phys.Rev.D**53** (1996) 6875.

[7] C. Correia da Silva and Ruth Williams, "Simplicial minisuperspace models", [gr-qc/9903003] . To be published in Class.Quantum Gravity.

[8] H.W.Hamber and R.M.Williams, Nucl.Phys.B**415** (1994) 463.